MULTIDIMENSIONAL NUMBERS AND SEMANTIC NUMERATION SYSTEMS: THEORETICAL FOUNDATION AND APPLICATION

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Abstract: In this article, we present a new class of numeration systems, namely Semantic Numeration Systems. The methodological background and theoretical foundations of such systems are considered. The concepts of abstract entity, entanglement and valence of abstract entities, and the topology of the numeration system are introduced. The proposed classification of semantic numeration systems allows to choose the numeration system depending on the problem being solved. Examples of the use of a two-dimensional number system for image compression problems and computation of a two-dimensional convolution are given.

Keywords: Semantics, Abstract Entity, Entanglement, Numeration System.

INTRODUCTION

The modern world is characterized as an ever-increasing volume of stored and processed information, as the growing diversity and complexity of algorithms for its processing. A key role in such areas as the theory of formal languages and automata, control systems and artificial intelligence systems, cryptography and image processing belongs to digital data processing. Along with the search for new algorithms of the digital processing, the search for effective systems for representing numbers and operating with them (i.e. numeration systems) has begun.

From systems with non-natural bases, combinatorial and Fibonacci [5, 6] to numeration systems with double and multiple bases [4] - this is the range of theoretical searches and attempts of application. The generalized presentation of these efforts was found in the concept of abstract number systems [1, 8] as an infinite regular language over a totally ordered alphabet. However, all these systems have, one way or another, one basic object of speculation - the basis of the numeration system.

Is there a way to "look" at a numeration system in a different way? What else, besides the base, can become a generatrix of the numeration system? Are there other number systems, except systems of natural, integer, rational, real and complex numbers? And if so, what could be their representation systems? Where and how can this be applied?

METAPHYSICS OF NUMBER

In modern mathematics the concept of a number is considered as basic, intuitively clear and not exactly defined. Only the main purpose of numbers is indicated: counting and ordering.

Perhaps, the first (and, as far as the author knows, the only) scientist who gave the definition of the concept of number (though not mathematical, but philosophical) is A.F. Losev. In [7] he points out that "a number is a definite form, or type, of pure semantic positing...". One is a steady of the abstract act of positing. The homogeneity of acts of positing determines the uniformity of the ones that make up the number. Those the key moment in the set of

natural numbers is the equipoise - "it is the same always and everywhere" [7].

However, the number is not just a collection of ones. According to A.F. Losev "the number is nothing other than a definite totality of elements". Let us give the phrase "definitely totality" a modern vision. We present the following definition of a number:

The number is the steady of the entanglement of pure semantic positing acts.

The formalization of the collection of ones in a number is realized precisely by the abstract act of entangled them into one whole. We will call this "act-glueing" as semantic entanglement and denote it as $\diamond$.

Semantic entanglement is a mental attitude in which the state of two or more objects (entities) must be described in a *semantic relationship* with each other (as one), even if the individual objects (entities) are not related to each other physically by the relations of generation, inclusion, interaction etc.

Only the semantically entangled ones form the cardinal characteristic of a set of ones, i.e. number as a pure quantity:

$$N = card (\{1, 1, ..., 1\}) = \diamond(1, 1, ..., 1).$$

The limit of semantic entanglement of ones in the "whole" on the m-th act of positing leads to the stopping of the process of further generation of "next" numbers and fixation of *another one*: 1_m, i.e. m-ki. This possibility of the other ones positing also makes it possible to construct positional numeration systems. One is always a semantic one of some (abstract) entity. In positional numeration systems, entities form the positions.

Let us introduce the concept of the (cardinal) Abstract Entity (Æ), the basic properties of which will be the ability to take it essence in discrete portions (units), accumulate them, keep them, and in the case of excess, transfer the result of overflow to another Abstract Entities that entangled with it:

$$Æ_i = (i, \ n, \ P \ | \ \alpha \vee \omega, \ q, \ p),$$

where *i* is the name (identifier) of Æi; *n* - cardinal capacity of Æ (the threshold of accumulation of entity ones 1i); α is the number of ones contained in Æ, within its capacity ($\alpha < n$); ω - an overflow - the number of ones equal to or greater than the Æ capacity ($\omega \geq n$); *q* is the number of ones arriving at the Æi input from the connected Æj; *p* is the over-

flow transfer value; *P* is the P-rule for determining the overflow transfer value (carry). Here $\forall$ n, α, ω, q, p $\in N_0$.

We call the "constant" part $Æ_i$ = (i, n, P) *the signature* of $Æ_i$, and the "variable" ($\alpha \vee \omega$, q, p) as its *state*.

The semantic postulate of positional numeration systems

Proposition 1. *The fact of overflowing of some entity $Æ_i$ makes sense for another entity $Æ_j$.*

On the assumption of the accepted methodological setting, it makes sense to talk about the directional semantic entanglement of abstract entities, in this case - the positions of the numeration system.

The directional semantic entanglement of abstract entities $Æ_i \rhd Æ_j$ or $Æ_j$ = Ent ($Æ_i$) here means that the result of the overflow of the entity $Æ_i$ in the form of the number p_i (carry) becomes the one(-s) 1_j of the entity $Æ_j$ in a meaningful way. In this case, the fact of carry formation in the entity $Æ_i$ coincides with the fact of the assumption of the one 1_j in the entity $Æ_j$ that is semantically entangled with $Æ_i$.

Thus, any positional numeration system (PNS) is a collection of direct entangled abstract entities of a given signature:

$$PNS = \rhd Æ_\bullet = Ent(...Ent(Æ_i)...).$$

Multinumbers and Polynumbers

The assumption of the heterogeneity of acts of positing with subsequent entanglement leads to the concept of a *multiset*, or a set with repeating elements. In other words, the multiset contains semantically entangled entities that have independent cardinals:

$$MM = \diamond(a \ |\#a; \ b \ |\#b; ...; \ c \ |\#c).$$

If we introduce certain m-ary multirelations on the set of elements and assume the multiplicity of these elements, then we will speak of *polysets* [2].

Example. A = (one blue cube, eleven red pyramids, three black spheres, seven green cones, and five black cones).

A system of semantically entangled cardinals of multisets will be called *multinumbers*, and of polysets are called *polynumbers*. For polynumbers in [2, 3], the concept of a multidimensional natural number was introduced and the Peano system of such numbers was justified. What kind of number system should provide repre-

sentation and account of heterogeneous semantically entangled entities?

Let us move on to **semantic numeration systems**.

It is in the semantic numeration system the concept of directed entanglement of heterogeneous Abstract Entities arises due to the semantic entanglement of heterogeneous acts of positing.

Proposition 2. *The fact of overflow of some entity $Æ_i$ is meaningful for (several) other entities $Æ_j$ ..., $Æ_k$.*

$$Æ_k$$
$$Æ_i \quad Æ_j \; \triangle\triangleright \text{or } Æ_j = Ent(Æ_i) \wedge Æ_k = Ent(Æ_i).$$

Proposition 3. *For some entity $Æi$, the facts of overflowing (several) other entities $Æj$, ..., $Æk$ are meaningful.*

$$Æ_k \triangleright Æ_i \triangle \quad \text{ili } Æ_i = Ent(Æ_j) \wedge Æ_i = Ent(Æ_k).$$

$$Æ_j$$

Thus, in the general case, the abstract entity can both "perceive" the results of the overflow of *m* other entities, and "transfer" the result of its overflow to other abstract entities. The number of $Æ$s, adhered to a given $Æ_i$ by its input, will be called the *passive valence* of $Æ_i$ and will be denoted as W_i. The number of $Æ$s, adhered to a given $Æ_i$ by its output, will be called the *active valence* of $Æ_i$ and will be denoted as V_i.

For any $Æ$ as the position of the semantic numeration system, now:

$$Æ_i = (\mathbf{i} \mid n_i, W_i, V_i, P_i \mid \alpha_i \vee w_i, q_i, p_i),$$

where *i* is the name (identifier) of $Æ$ - in general, a tuple of partial names (characters, numbers) that constitute the full name of $Æ_i$; n_i - the capacity of $Æ_i$ (or the threshold); W_i is the passive valence; V_i is the active valence; α_i is the number of ones contained in $Æ_i$, within its capacity ($\alpha_i < n_i$); w_i - overflow ($w_i \geq n_i$); q_i - the number of units of the entity that have simultaneously entered the input $Æ_i$ from the others $Æ$s connected to it; p_i is the overflow transfer value (carry). P_i is the P-rule for $Æ_i$.

The choice of $Æ$s, their signatures and the method of the formation of a numeration system from them should be determined precisely by the given semantic interaction of entities within the framework of the problem being solved. The structure of the numeration system in accordance with the specified

interaction semantics of $Æ$s will be called the *topology of valence entanglement* (or valent entanglement matrix - VEM) of semantic numeration system (SNS). Thus, the SNS description will consist of a signature:
Signt (SNS) = <I, i. | n., [VEM], P.>,
and a state:
State (SNS) = < α. $\vee$ ω., q., p. >

Classification of semantic numeration systems

By the variability of the structure (topology of valence entanglement):
- constants;
- variables: - functionally defined / assigned;
                  - uncertain (random, fuzzy).

By the regularity of the structure:
- regular, i.e. invariant to the structural shift (for example, a 2D lattice);
- irregular.

By the variability of the abstract entities signatures:
• By directions:
  - isotropic (identical in all directions);
  - anisotropic (different in different directions).
• Inside each direction:
  - homogeneous ($n_i (dir_l) = const$);
  - heterogeneous ($n_i (dir_l) = var$);
  - mixed (homogeneous in one direction and heterogeneous in others).

By type of valence:
- isovalent ($W_i = V_i$, "$Æ_i$);
- heterovalent ($\$Æ_i, W_i \neq V_i$);

By type of P-rule:
- standard numerical (p = [ω / n], α` = ω mod (n));
- special (p = f (ω, n, i)).

On stability:
- stable (the procedure for representing a multi-number in SNS ends in a finite number of steps);
- unstable (the procedure is infinite).

By the controllability of the signature:
- autonomous (not controlled, hard-set);
- controlled (adaptive or under external control).

Along with the representation of numbers, any numeration system should provide the realization of elementary arithmetic operations with them. For SNS with a regular structure (2D lattice) in [2], operations of addition and multiplication of two-dimensional numbers are justified. A one-to-one

correspondence between the sum and products of multidimensional numbers and their representations in the 2D lattice SNS is proved.

Applications
1. Compression of black and white images.

The main idea of the proposed method of image (or fragment) compression is to give the digital relief of the image the sense of the representation of the polynumber A in a regular isotropic SNS with $n_{ij} = 2$ and $W_{ij} = V_{ij} = 2$, "$Æ_{ij}$. By the inverse transformation from the given representation, we get the polynumber A as a more compact numerical set intended for storage or transmission. Restoration of the image (decompression) consists in making the procedure for representing the polynumber A in the same SNS.
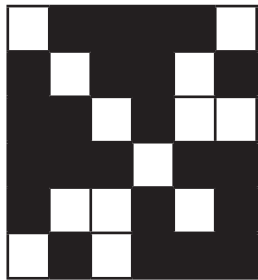
$$N_2 = 23 \rightarrow \langle 23 \rangle_{00} = A$$

Block diagram of a communication system based on a new data compression principle.
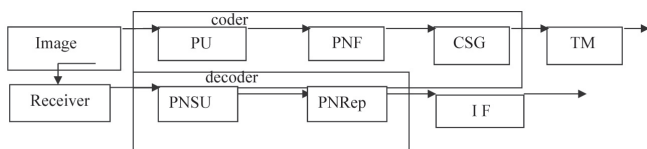
In the diagram: PU - partition unit, PNF - polynumber former, CSG - code sequence generator, TM- transmitter, PNSU – polynumber selection unit, PNRep – polynumber representation, IF – image former.

To compress halftone images, it is necessary to use a regular isotropic SNS with a base equal to the number of gray gradations.

Advantages of the proposed approach to data compression are:

1. The possibility of implementing progressive data(image) compression.
2. Potentially high compression ratio.
3. The possibility of lossless data compression.
4. The simplicity of the decoding (restoring) data algorithm on the receiving side.
5. The possibility of adaptive regulation of the transmitted (decoded) information volume depending on the permissible level of losses.

2. The method of calculating the two-dimensional convolution.

The basis for digital information processing is computational algorithms for convolution and discrete Fourier transform. The calculation of convolutions (one-dimensional and multidimensional) is currently carried out with the help of discrete Fourier transform algorithms, polynomial-theoretic and number-theoretic transformations. The main disadvantages of the Fourier transform are: the use of transcendental functions (sin and cos), the use of complex arithmetic even with real convolution. It leads to a doubling of the numerical fields dimension. Linear two-dimensional convolution in general form is expressed as

$$\sum \sum h(m_1, m_2)\, x(r_1\text{-}m_1, r_2\text{-}m_2) = y(r_1, r_2),$$

where $x(r_1\text{-}m_1, r_2\text{-}m_2)$ is the input two-dimensional sequence; h (m1, m2) is the two-dimensional impulse response of the system; y (r1, r2) is the output two-dimensional sequence.

We make use of the formal correspondence between the operations of two-dimensional convolution and the multiplication of two multinomials, on the one hand, and the one-to-one correspondence of products of multidimensional numbers (multinumbers and polynumbers) and their representations on the other. Then the calculation of the two-dimensional convolution can be performed in the 2D lattice SNS, carrying out instead of the discrete Fourier transform the transformation "2D representation of the number → polynumber", multiply the input polynomial and the impulse response, and then apply the transformation "resulting polynumber → 2D-representation of the result polynumber".

Advantages of the proposed method:

1. It does not require the use of complex quantities (spaces);
2. It does not use harmonic or special functions for the transformation;
3. It allows you to replace complex functional transformations with arithmetic ones;
4. Simplicity and clarity.

Conclusion

The Semantic Numeration Systems theory is at the initial stage of its development. Nevertheless, even now it can be assumed that SNS will be in demand in many areas related to the digital processing, among which are the following:

- cryptoprotection - the creation of fundamentally new cryptosystems to protect information of increased cryptographic strength;
- computer databases - compact representation, efficient storage and fast data transfer (exchange);
- geoinformation systems (GIS) - compact storage of digital terrain maps, their efficient transmission through communication channels;
- biometrics - effective identification of a person by fingerprints, the iris of the eye, photographs;
- medical technologies (tomography) - fundamentally new algorithms for 3D reconstruction;
- radars, sonars, and radio navigation - high-speed data processing;
- radio communication, including mobile communication - increasing the bandwidth of communication channels.

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